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A derivation of an off-shell 2D, $\mathcal{N}$ = (2, 2) supergravity chiral projection operator

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Abstract
Utilizing the known off-shell formulation of 2D, $\mathcal{N}$ = (2, 2) supergravity containing a finite number of auxiliary fields, there is shown to exist a simple form for a 'chiral projection operator' and an explicit expression for it is given.

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1. Introduction

Some years ago, the first discussion of a 2D, $\mathcal{N}$ = (2, 2) supergravity theory was given [1] and elaborated upon in a subsequent work [2]. In both of these works, no discussion of the auxiliary fields required to close the local supersymmetry algebra without resorting to a set of equations of motion was undertaken. Finally, 13 years after the first such discussion, a formulation of 2D, $\mathcal{N} = 4$ supergravity theory that included a *finite* number of auxiliary fields was presented [3]. The relation between these on-shell versus off-shell supergravity theories is not as direct as one might imagine. The reason for this is the existence of a great plethora of 2D, $\mathcal{N} = 4$ scalar multiplets [4]. Previous experience has shown that when this situation exists, the diverse scalar multiplets foreshadow distinct formulations of correspondingly diverse off-shell supergravity theories.

The existence of a relatively simple off-shell formulation of a 2D, $\mathcal{N} = 4$ supergravity theory implies that there should exist a straightforward way to completely develop an *efficient* local integration theory for the associated local Salam–Strathdee superspace. In this work, we will take the first major step in this direction by providing the initial discussion of a local 2D, $\mathcal{N} = 4$ chiral projection operator.

2. A review of an off-shell 2D, $\mathcal{N}$ = (2, 2) superspace supergravity geometry

Let us begin by reviewing the results in [3]. This work showed that there exist component fields $\left(e_a{}^m, \psi_a{}^{\alpha i}, A_{ai}{}^j, B, G, H\right)$ which describe an off-shell 2D, $\mathcal{N}$ = (2, 2) supergravity theory. These are the components that remain after imposing the following constraints on the 2D, $N = 4$ superspace supergravity covariant derivative, (with $\phi_{\alpha\beta} \equiv -\mathrm{i}[C_{\alpha\beta} G + \mathrm{i}(\gamma^3)_{\alpha\beta} H]$)

$$\begin{aligned}
[\nabla_{\alpha i}, \nabla_{\beta j}\} &= 2\overline{B}[C_{\alpha\beta} C_{ij} \mathcal{M} - (\gamma^3)_{\alpha\beta} \mathcal{Y}_{ij}],\\
[\nabla_{\alpha i}, \overline{\nabla}_\beta{}^j\} &= 2\big[\mathrm{i}\delta_i{}^j (\gamma^c)_{\alpha\beta} \nabla_c + \delta_i{}^j \phi_\alpha{}^\gamma (\gamma^3)_{\gamma\beta} \mathcal{M} - \mathrm{i}\phi_{\alpha\beta} \mathcal{Y}_i{}^j\big],\\
[\nabla_{\alpha i}, \nabla_b\} &= \mathrm{i}\tfrac{1}{2}\phi_\alpha{}^\gamma (\gamma_b)_\gamma{}^\beta \nabla_{\beta i} + \mathrm{i}\tfrac{1}{2}(\gamma^3\gamma_b)_\alpha{}^\beta \overline{B} C_{ij} \overline{\nabla}_\beta{}^j\\
&\quad - \mathrm{i}(\gamma^3\gamma_b)_{\alpha\beta} \overline{\Sigma}^\beta{}_i \mathcal{M} + \mathrm{i}(\gamma_b)_{\alpha\beta} \overline{\Sigma}^\beta{}_j \mathcal{Y}_i{}^j,\\
[\nabla_a, \nabla_b\} &= -\tfrac{1}{2}\epsilon_{ab}\big[(\gamma^3)_\alpha{}^\beta \Sigma^{\alpha i} \nabla_{\beta i} + (\gamma^3)_\alpha{}^\beta \overline{\Sigma}^\alpha{}_i \overline{\nabla}_\beta{}^i + \mathcal{R}\mathcal{M} + i\mathcal{F}_i{}^j \mathcal{Y}_j{}^i\big].
\end{aligned} \tag{1}$$

The consistency of the Bianchi identities constructed from the commutator algebra above required the conditions,

$$\begin{aligned}
\overline{\nabla}_\alpha{}^i B &= 0, \qquad \nabla_{\alpha i} B = -2C_{ij}(\gamma^3)_{\alpha\beta} \Sigma^{\beta j},\\
\nabla_{\alpha i} G &= \overline{\Sigma}_{\alpha i}, \qquad \nabla_{\alpha i} H = \mathrm{i}(\gamma^3)_\alpha{}^\beta \overline{\Sigma}_{\beta i},\\
\overline{\nabla}_\alpha{}^i \Sigma^{\beta j} &= \mathrm{i}C^{ij} (\gamma^3\gamma^a)_\alpha{}^\beta \nabla_a B,\\
\nabla_{\alpha i} \Sigma^{\beta j} &= \tfrac{1}{2}\delta_\alpha{}^\beta \delta_i{}^j [\mathcal{R} - 2G^2 - 2H^2 - 2B\overline{B}] + \mathrm{i}(\gamma^3)_\alpha{}^\beta \mathcal{F}_i{}^j\\
&\quad + \mathrm{i}\tfrac{1}{2}\delta_i{}^j (\gamma^a)_\alpha{}^\beta (\nabla_a G) - \tfrac{1}{2}\delta_i{}^j (\gamma^3\gamma^a)_\alpha{}^\beta (\nabla_a H).
\end{aligned} \tag{2}$$

The component gauge fields occur in the above supertensors in the following manner.

$$\begin{aligned}
\mathcal{R}| &= \epsilon^{ab}\big\{\mathcal{R}_{ab}(\hat{\omega}) + \big[\mathrm{i}2(\gamma^3\gamma_a)_{\alpha\beta} \psi_b{}^{\alpha i} \overline{\Sigma}^\beta{}_i + \text{h.c.}\big]\\
&\quad + 4\phi_\alpha{}^\gamma (\gamma^3)_{\gamma\beta} \psi_a{}^{\alpha i} \overline{\psi}_b{}^\beta{}_i - 2\big[C_{ij} \overline{B} \psi_a{}^{\alpha i} \psi_{b\alpha}{}^j + \text{h.c.}\big]\big\},\\
\Sigma^{\alpha i}| &= \epsilon^{ab}\big\{\psi_{ab}{}^{\beta i} (\gamma^3)_\beta{}^\alpha - \mathrm{i}\psi_a{}^{\beta i} \phi_\beta{}^\gamma (\gamma^3\gamma_b)_\gamma{}^\alpha + \mathrm{i}C^{ij} B \overline{\psi}_a{}^\beta{}_j (\gamma_b)_\beta{}^\alpha\big\},\\
\mathcal{F}_i{}^j\big| &= \epsilon^{ab}\big\{\mathrm{F}_{ab}(A)_i{}^j - \mathrm{i}2(\gamma_a)_{\alpha\beta}\big[\psi_b{}^{\alpha j} \overline{\Sigma}^\beta{}_i + \overline{\psi}_b{}^\alpha{}_i \Sigma^{\beta j} - \tfrac{1}{2}\delta_i^j \big(\psi_b{}^{\alpha k} \overline{\Sigma}^\beta{}_k + \overline{\psi}_b{}^\alpha{}_k \Sigma^{\beta k}\big)\big]\\
&\quad - 4\phi_{\alpha\beta}\big[\psi_a{}^{\alpha j} \overline{\psi}_b{}^\beta{}_i - \tfrac{1}{2}\delta_i^j \psi_a{}^{\alpha k} \overline{\psi}_b{}^\beta{}_k\big]\\
&\quad - 2(\gamma^3)_{\alpha\beta}\big[\overline{B}\big(C_{ik} \psi_a{}^{\alpha k} \psi_b{}^{\beta k} - \tfrac{1}{2}\delta_i^j C_{kl} \psi_a{}^{\alpha k} \psi_b{}^{\beta l}\big)\\
&\quad + B\big(C^{jk} \overline{\psi}_a{}^\alpha{}_i \overline{\psi}_b{}^\beta{}_k - \tfrac{1}{2}\delta_i^j C^{kl} \overline{\psi}_a{}^\alpha{}_k \overline{\psi}_b{}^\beta{}_l\big)\big]\big\},
\end{aligned} \tag{3}$$

where $\epsilon^{ab}\mathcal{R}_{ab}(\hat{\omega})$ is the usual two-dimensional curvature in terms of $\mathrm{e}_a{}^m$ and $\hat{\omega}_m$.

3. 2D, $\mathcal{N}$ = (2, 2) local superspace integration and chiral projector

In a rigid superspace (i.e. in a 'flat supergravity background'), the derivation of component results follows most efficiently [5] from replacing the integration of fermionic coordinates by a process using *first* application of the superspace covariant derivative followed by taking a limit in which all Grassmann coordinates are taken to zero. For the discussion of this communication, this amounts to the validity of the following equation,

$$\begin{aligned}
S = \int \mathrm{d}^2\sigma\, \mathrm{d}^4\theta\, \mathrm{d}^4\overline{\theta}\mathcal{L} &= \lim_{\theta\to 0} \int \mathrm{d}^4\sigma \frac{1}{2}[\mathrm{D}^4\overline{\mathrm{D}}^4\mathcal{L} + \text{h.c.}]\\
&\equiv \int \mathrm{d}^2\sigma \frac{1}{2}[\mathrm{D}^4\overline{\mathrm{D}}^4\mathcal{L} + \text{h.c.}]|
\end{aligned} \tag{4}$$

In the presence of a supergravity background the integration over the superspace measure is modified by the insertion of the supergravity vielbein

$$\int \mathrm{d}^2\sigma\, \mathrm{d}^4\theta\, \mathrm{d}^4\overline{\theta} \to \int \mathrm{d}^2\sigma\, \mathrm{d}^4\theta\, \mathrm{d}^4\overline{\theta}\mathrm{E}^{-1} \tag{5}$$

and this in turn implies a modification of (5) to the form [6]

$$S = \int d^2\sigma \frac{1}{2} e^{-1}[\mathcal{D}^{(4)}\overline{\mathcal{D}}^{(4)}\mathcal{L} + \text{h.c.}]| \tag{6}$$

which is written in terms of two differential operators, $\mathcal{D}^{(4)}$ and $\overline{\mathcal{D}}^{(4)}$. The first of these is called the 'density projection operator' and the second is called 'chiral projection operator'. The appearance of these two distinct operators is characteristic of any superspace in which chirality has a well defined meaning.

As outlined in [6], the density projection operator must be of the form

$$\mathcal{D}^{(4)} = \sum_{i=0}^{4} b_{(4-i)} \cdot [(\nabla) \times \cdots \times (\nabla)^{4-i}], \tag{7}$$

in terms of some field-dependent coefficients $b_{(4-i)}$ and powers of the spinorial superspace supergravity covariant derivative $\nabla_{\alpha i}$. In the work of [5], a 'handicraft' method for finding these coefficients was described. The works of [6] describe more powerful methods for deriving this operator (which will be a topic of future efforts). In a similar manner, the chiral projection operator must be of the form

$$\overline{\mathcal{D}}^{(4)} = \sum_{i=0}^{4} a_{(4-i)} \cdot [(\overline{\nabla}) \times \cdots \times (\overline{\nabla})^{4-i}], \tag{8}$$

in terms of some field-dependent coefficients $a_{(4-i)}$ (that are distinct from the $b_{(4-i)}$ coefficients) and powers of the spinorial superspace supergravity covariant derivative $\overline{\nabla}^i_\alpha$. Since the two sets of coefficients are distinct it follows that $\mathcal{D}^{(4)}$ is not the conjugate of $\overline{\mathcal{D}}^{(4)}$.

Here we give our main result for the local 2D, $\mathcal{N} = (2, 2)$ superspace described by (1)

$$\begin{aligned} S &= \int d^2\sigma\, d^4\theta\, d^4\overline{\theta} E^{-1}\mathcal{L} \\ &= \int d^2\sigma\, d^4\theta \mathcal{E}^{-1}\frac{1}{2}[\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta}\mathcal{L}\big| + \text{h.c.} \\ &= \int d^2\sigma\, e^{-1}\frac{1}{2}\mathcal{D}^{(4)}_{\mathcal{P}}[\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta}\mathcal{L}\big| + \text{h.c.} \end{aligned} \tag{9}$$

where on the first line E^{-1} is the density factor (i.e. the superdeterminant of the vielbein of the full 2D, $\mathcal{N} = (2, 2)$ superspace) and on the second line of this equation $\mathcal{E}^{-1}$ is the chiral density factor. The superdifferential operator

$$\overline{\mathcal{D}}^{(4)} = [\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta} \tag{10}$$

is the 2D, $\mathcal{N} = (2, 2)$ chiral projection operator. The main new result we have to report is its explicit form which satisfies

$$\overline{\nabla}^i_\gamma\overline{\mathcal{D}}^{(4)}\Psi = \overline{\nabla}^i_\gamma[\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta}\Psi = 0 \tag{11}$$

for any general scalar superfield Ψ.

Upon comparing (8) with (10), we note that the coefficients for the former can be read from the latter and imply that

$$\begin{aligned} &a_{(0)} = 0, \qquad a_{(1)} = 0, \qquad a_{(2)} = -2B(\gamma^3)^{\alpha\beta}C_{ij}, \\ &a_{(3)} = 0, \qquad a_{(4)} = \tfrac{1}{2}C_{ij}C_{kl}[C^{\alpha\gamma}C^{\beta\delta} + C^{\alpha\delta}C^{\beta\gamma}]. \end{aligned} \tag{12}$$

So that explicitly we have

$$\overline{\mathcal{D}}^{(4)} = \tfrac{1}{2}C_{ij}C_{kl}[C^{\alpha\gamma}C^{\beta\delta} + C^{\alpha\delta}C^{\beta\gamma}]\overline{\nabla}^i_\alpha\overline{\nabla}^j_\beta\overline{\nabla}^k_\gamma\overline{\nabla}^l_\delta - 2B(\gamma^3)^{\alpha\beta}C_{ij}\overline{\nabla}^i_\alpha\overline{\nabla}^j_\beta. \tag{13}$$

Even without knowing the explicit form of the density projection operator, (9) implies

$$\int d^2\sigma \, d^4\theta \, d^4\overline{\theta} \mathrm{E}^{-1} = 0, \tag{14}$$

i.e. the superspace described by (1) has a vanishing supervolume. The derivation of these results is described in the appendix.

4. Conclusion

With this present work, we have completed half of the task of developing an efficient local superspace integration theory for two-dimensional theories that possess eight supercharges. The crux of this presentation was the unveiling of the explicit form of the chiral projection operator given in (11).

> *'Everone takes the limits of his own vision for the limits of the world'*—Arthur Schopenhauer.

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Appendix A. Definitions and conventions

For two-dimensional superspaces, we use the following conventions for the quantities associated with spinors.

$$\begin{aligned} &\eta_{ab} = (1, -1), \qquad \epsilon_{ab}\epsilon^{cd} = -\delta_{[a}{}^{c}\delta_{b]}{}^{d}, \qquad \epsilon^{01} = +1, \\ &(\gamma^a)_\alpha{}^\gamma(\gamma^b)_\gamma{}^\beta = \eta^{ab}\delta_\alpha{}^\beta - \epsilon^{ab}(\gamma^3)_\alpha{}^\beta. \end{aligned} \tag{A.1}$$

The last one of these relations imply

$$\gamma^a\gamma_a = 2\mathbf{I}, \qquad \gamma^3\gamma^a = -\epsilon^{ab}\gamma_b. \tag{A.2}$$

Some useful Fierz identities are as follows:

$$\begin{aligned} &C_{\alpha\beta}C^{\gamma\delta} = \delta_{[\alpha}{}^{\gamma}\delta_{\beta]}{}^{\delta}, \\ &(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} + (\gamma^3)_{\alpha\beta}(\gamma^3)^{\gamma\delta} = -\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta}, \\ &(\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} + (\gamma^3)_{(\alpha}{}^{\gamma}(\gamma^3)_{\beta)}{}^{\delta} = \delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta}, \\ &(\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} = -2(\gamma^3)_{\alpha\beta}(\gamma^3)^{\gamma\delta}, \\ &2(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} + (\gamma^3)_{(\alpha}{}^{\gamma}(\gamma^3)_{\beta)}{}^{\delta} = -\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta}, \\ &(\gamma_a)_\alpha{}^\delta\delta_\beta{}^\gamma + (\gamma^3\gamma_a)_\alpha{}^\gamma(\gamma^3)_\beta{}^\delta = (\gamma^3\gamma_a)_{\alpha\beta}(\gamma^3)^{\gamma\delta}. \end{aligned} \tag{A.3}$$

In terms of an explicit representation, we can define the 2D γ-matrices in terms of the usual Pauli matrices according to

$$(\gamma^0)_\alpha{}^\beta \equiv (\sigma^2)_\alpha{}^\beta, \qquad (\gamma^1)_\alpha{}^\beta \equiv -\mathrm{i}(\sigma^1)_\alpha{}^\beta, \qquad \gamma^3 \equiv (\sigma^3)_\alpha{}^\beta. \tag{A.4}$$

As can be seen, these satisfy the second line in (A.1). The spinor metric $C_{\alpha\beta}$ and its inverse $C^{\alpha\beta}$ can be identified as

$$C_{\alpha\beta} \equiv (\sigma^2)_{\alpha\beta}, \qquad C^{\alpha\beta} \equiv -(\sigma^2)^{\alpha\beta}. \tag{A.5}$$

Using this explicit representation, it is easy to show the following symmetry properties

$$\begin{aligned} &(\gamma^a)_{\alpha\beta} = (\gamma^a)_{\beta\alpha}, \qquad (\gamma^3)_{\alpha\beta} = (\gamma^3)_{\beta\alpha}, \qquad C_{\alpha\beta} = -C_{\beta\alpha}, \\ &(\gamma^a)^{\alpha\beta} = (\gamma^a)^{\beta\alpha}, \qquad (\gamma^3)^{\alpha\beta} = (\gamma^3)^{\beta\alpha}, \qquad C^{\alpha\beta} = -C^{\beta\alpha}. \end{aligned} \tag{A.6}$$

In a similar manner the following complex conjugation properties can be derived

$$\left[(\gamma^a)_\alpha{}^\beta\right]^* = -(\gamma^a)_\alpha{}^\beta, \qquad [(\gamma^3)_\alpha{}^\beta]^* = +(\gamma^3)_\alpha{}^\beta, \tag{A.7}$$

$$\begin{aligned} &[(\gamma^a)_{\alpha\beta}]^* = (\gamma^a)_{\alpha\beta}, \qquad [(\gamma^3)_{\alpha\beta}]^* = -(\gamma^3)_{\alpha\beta}, \qquad [C_{\alpha\beta}]^* = -C_{\alpha\beta}, \\ &[(\gamma^a)^{\alpha\beta}]^* = (\gamma^a)^{\alpha\beta}, \qquad [(\gamma^3)^{\alpha\beta}]^* = -(\gamma^3)^{\alpha\beta}, \qquad [C^{\alpha\beta}]^* = -C^{\alpha\beta}. \end{aligned} \tag{A.8}$$

Due to the first relation in (A.8) we see that this choice of gamma matrices is in a Majorana representation and thus the simplest spinors such as $\psi^\alpha(x)$, may be chosen to be real, i.e.

$$[\psi^\alpha(x)]^* = \psi^\alpha(x), \tag{A.9}$$

and we can raise and lower spinor indices according to

$$\psi^\alpha(x) = C^{\alpha\beta}\psi_\beta(x), \qquad \psi_\alpha(x) = \psi^\beta(x)C_{\beta\alpha}. \tag{A.10}$$

It can be seen as a consequence that

$$[\psi_\alpha(x)]^* = -\psi_\alpha(x). \tag{A.11}$$

Of course, it is always possible to introduce complex spinors also.

The two generators that define the holonomy group of the 2D, $\mathcal{N} = (2, 2)$ superspace are $\mathcal{M}$ and $\mathcal{Y}_i{}^j$, respectively for the 2D Lorentz group (SL(2,R)) and an internal SU(2) group. These are defined to act as

$$[\mathcal{M}, \nabla_{\alpha i}] = \tfrac{1}{2}(\gamma^3)_\alpha{}^\beta\nabla_{\beta i}, \qquad \left[\mathcal{Y}_i{}^j, \nabla_{\alpha k}\right] = \delta_k{}^j\nabla_{\alpha i} - \tfrac{1}{2}\delta_i{}^j\nabla_{\alpha k}. \tag{A.12}$$

The rules for manipulating the SU(2) spinors are very much similar to the ones used for the SL(2,R) spinor indices. The SU(2) metric C_{ij} and its inverse C^{ij} can be identified as

$$C_{ij} \equiv (\sigma^2)_{ij}, \qquad C^{ij} \equiv -(\sigma^2)^{ij}, \tag{A.13}$$

so that

$$C_{ij} = -C_{ji}, \qquad C^{ij} = -C^{ji}, \qquad C_{ij}C^{kl} = \delta_i{}^k\delta_j{}^l - \delta_i{}^l\delta_j{}^k. \tag{A.14}$$

We raise and lower SU(2) indices according to

$$\psi^i(x) = C^{ij}\psi_j(x), \qquad \psi_\iota(x) = \psi^j(x)C_{ji}, \tag{A.15}$$

that are directly the analogs for raising and lowering indices on SL(2,R) tensors.

Appendix B. Derivation of the chiral projector

In this appendix, we give a presentation that leads to the form of the chiral projector.

The expressions in (1) give the commutator algebra of the 4D, $\mathcal{N} = (2, 2)$ supergravity derivative. However, in order to derive the chiral projection formula we need to go beyond that algebra. For this purpose, we introduce new second-order differential operators denoted by $\nabla^{(2)}_{\alpha\beta}$ and $\nabla^{(2)}_{ij}$ and defined by the equations

$$\begin{aligned} \nabla^{(2)}_{\alpha\beta} &= \tfrac{1}{2}C^{ij}[\nabla_{\alpha i}\nabla_{\beta j} + \nabla_{\beta i}\nabla_{\alpha j}], \\ \nabla^{(2)}_{ij} &= \tfrac{1}{2}C^{\alpha\beta}[\nabla_{\alpha i}\nabla_{\beta j} + \nabla_{\alpha j}\nabla_{\beta i}]. \end{aligned} \tag{B.1}$$

Using these definitions implies

$$\begin{aligned}\nabla_{\alpha i}\nabla_{\beta j} &= \tfrac{1}{2}[\nabla_{\alpha i}, \nabla_{\beta j}] + \tfrac{1}{2}\{\nabla_{\alpha i}, \nabla_{\beta j}\},\\ &= \tfrac{1}{2}C_{ij}\nabla^{(2)}_{\alpha\beta} + \tfrac{1}{2}C_{\alpha\beta}\nabla^{(2)}_{ij} + \tfrac{1}{2}\{\nabla_{\alpha i}, \nabla_{\beta j}\},\\ &= \tfrac{1}{2}C_{ij}\nabla^{(2)}_{\alpha\beta} + \tfrac{1}{2}C_{\alpha\beta}\nabla^{(2)}_{ij} + \overline{B}[C_{\alpha\beta}C_{ij}\mathcal{M} - (\gamma^3)_{\alpha\beta}\mathcal{Y}_{ij}].\end{aligned} \tag{B.2}$$

Next we use (1), (B.1) and (B.2) to derive

$$\left[\nabla_{\alpha i}, \nabla^{(2)}_{\beta\gamma}\right] = -\tfrac{1}{4}\overline{B}\left[C_{\alpha(\beta}(\gamma^3)_{\gamma)}{}^{\delta} - 3(\gamma^3)_{\alpha(\beta}\delta_{\gamma)}{}^{\delta}\right]\nabla_{\delta i} - \overline{B}\nabla_{(\beta j}\left[C_{\alpha|\gamma)}\delta_i{}^j\mathcal{M} - (\gamma^3)_{\alpha|\gamma)}\mathcal{Y}_i{}^j\right]. \tag{B.3}$$

Using more manipulations there are found additional identities involving $\nabla^{(2)}_{\alpha\beta}$.

$$\nabla_{(\alpha|i}\nabla^{(2)}_{|\beta\gamma)} = 2\overline{B}(\gamma^3)_{(\alpha\beta}\left[\nabla_{\gamma)i} + \nabla_{\gamma)j}\mathcal{Y}_i{}^j\right], \tag{B.4}$$

$$\nabla^{(2)}_{\alpha\beta}\nabla^{(2)\beta\gamma} = \tfrac{1}{2}\delta_\alpha{}^\gamma\nabla^{(2)\beta\delta}\nabla^{(2)}_{\beta\delta} + \tfrac{1}{2}C^{\beta\delta}C^{\gamma\epsilon}\left[\nabla^{(2)}_{\alpha\beta}, \nabla^{(2)}_{\delta\epsilon}\right], \tag{B.5}$$

$$\begin{aligned}\left[\nabla^{(2)}_{\alpha\beta}, \nabla^{(2)}_{\delta\epsilon}\right] = {}& 2\overline{B}\left[(\gamma^3)_{\alpha\beta}\nabla^{(2)}_{\delta\epsilon} - (\gamma^3)_{\delta\epsilon}\nabla^{(2)}_{\alpha\beta}\right] + 4\overline{B}^2[C_{\alpha(\delta}(\gamma^3)_{\epsilon)\beta} + C_{\beta(\delta}(\gamma^3)_{\epsilon)\alpha}]\mathcal{M}\\ &- \overline{B}[C_{\alpha(\delta}\nabla^{(2)}{}_{\epsilon)\beta} + C_{\beta(\delta}\nabla^{(2)}{}_{\epsilon)\alpha}]\mathcal{M}\\ &+ \tfrac{1}{2}\overline{B}\nabla^{(2)}{}_{ij}[C_{\alpha(\delta}(\gamma^3)_{\epsilon)\beta} + C_{\beta(\delta}(\gamma^3)_{\epsilon)\alpha}]\mathcal{Y}^{ij}.\end{aligned} \tag{B.6}$$

With the identities of (B.4)–(B.6) in hand, the proof of (11) follows from the steps described below.

(1) The first step is to multiply (B.4) from the right by $\nabla^{(2)\beta\gamma}\overline{\Psi}$ which yields

$$\begin{aligned}&\nabla_{\alpha i}\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma}\overline{\Psi} + 2\nabla_{\gamma i}\nabla^{(2)}_{\alpha\beta}\nabla^{(2)\beta\gamma}\overline{\Psi} - 2\overline{B}(\gamma^3)_{\beta\gamma}\nabla_{\alpha i}\nabla^{(2)\beta\gamma}\overline{\Psi}\\ &\quad - 4\overline{B}(\gamma^3)_{\alpha\beta}\nabla_{\gamma i}\nabla^{(2)\beta\gamma}\overline{\Psi} = 0.\end{aligned} \tag{B.7}$$

(2) In the second term of (B.7), we substitute the identity from (B.5) to find

$$\begin{aligned}&2\nabla_{\alpha i}\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma}\overline{\Psi} + C^{\beta\delta}C^{\gamma\epsilon}\nabla_{\gamma i}\left[\nabla^{(2)}_{\alpha\beta}, \nabla^{(2)}_{\delta\epsilon}\right]\overline{\Psi}\\ &\quad - 2\overline{B}(\gamma^3)_{\beta\gamma}\nabla_{\alpha i}\nabla^{(2)\beta\gamma}\overline{\Psi} - 4\overline{B}(\gamma^3)_{\alpha\beta}\nabla_{\gamma i}\nabla^{(2)\beta\gamma}\overline{\Psi} = 0.\end{aligned} \tag{B.8}$$

(3) We next multiply the identity of (B.5) from the right by $\overline{\Psi}$ to find

$$\left[\nabla^{(2)}_{\alpha\beta}, \nabla^{(2)}_{\delta\epsilon}\right]\overline{\Psi} = 2\overline{B}\left[(\gamma^3)_{\alpha\beta}\nabla^{(2)}_{\delta\epsilon} - (\gamma^3)_{\delta\epsilon}\nabla^{(2)}_{\alpha\beta}\right]\overline{\Psi}. \tag{B.9}$$

(4) The substitution of the result on the left-hand side of (B.9) into the second term of (B.8) yields

$$\begin{aligned}&2\nabla_{\alpha i}\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma}\overline{\Psi} - 2\overline{B}(\gamma^3)^{\beta\gamma}\nabla_{\gamma i}\nabla^{(2)}{}_{\alpha\beta}\overline{\Psi}\\ &\quad - 2\overline{B}(\gamma^3)_{\beta\gamma}\nabla_{\alpha i}\nabla^{(2)\beta\gamma}\overline{\Psi} - 2\overline{B}(\gamma^3)_{\alpha\beta}\nabla_{\gamma i}\nabla^{(2)\beta\gamma}\overline{\Psi} = 0,\\ &2\nabla_{\alpha i}\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma}\overline{\Psi} + 2\overline{B}(\gamma^3)^{\gamma}{}_{\beta}\nabla_{\gamma i}\nabla^{(2)}{}_{\alpha}{}^{\beta}\overline{\Psi}\\ &\quad - 2\overline{B}(\gamma^3)_{\beta\gamma}\nabla_{\alpha i}\nabla^{(2)\beta\gamma}\overline{\Psi} - 2\overline{B}(\gamma^3)_{\alpha\beta}\nabla_{\gamma i}\nabla^{(2)\gamma\beta}\overline{\Psi} = 0.\end{aligned} \tag{B.10}$$

(5) The second and last terms on the final line in (B.10) added together equal to the third term in the same equation. Thus, we obtain the final result

$$\begin{aligned}&2\nabla_{\alpha i}\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma}\overline{\Psi} - 4\overline{B}(\gamma^3)_{\beta\gamma}\nabla_{\alpha i}\nabla^{(2)\beta\gamma}\overline{\Psi} = 0\\ &2\nabla_{\alpha i}\left(\nabla^{(2)}_{\beta\gamma}\nabla^{(2)\beta\gamma} - 2\overline{B}(\gamma^3)_{\beta\gamma}\nabla^{(2)\beta\gamma}\right)\overline{\Psi} = 0\\ &\nabla_{\alpha i}(\nabla^{(2)\beta\gamma} - 2\overline{B}(\gamma^3)^{\beta\gamma})\nabla^{(2)}{}_{\beta\gamma}\overline{\Psi} = 0.\end{aligned} \tag{B.11}$$

Upon taking the complex conjugate of this final equation in (B.11) the result of (11) follows.

References

[1] Admello M *et al* 1976 *Nucl. Phys.* B **114** 297
[2] Pernici M and van Nieuwenhuizen P 1986 *Phys. Lett.* B **169** 381
[3] Gates S J Jr, Liang Lu and Oerter Robert N 1989 *Phys. Lett.* B **218** 33
Gates S J Jr, Hassoun Y and van Nieuwenhuizen P 1989 *Nucl. Phys.* B **317** 302
[4] Gates S J Jr and Ketov S V 1998 *Phys. Lett.* B **418** 111 (arXiv:hep-th/hep-th/9504077)
Gates S J Jr and Ketov S V 1998 *Phys. Lett.* B **418** 119
Ivanov E and Sutulin A 2005 *Theor. Math. Phys.* **145** 1425
Ivanov E and Sutulin A 2005 *Teor. Mat. Fiz.* **145** 66 (arXiv:hep-th/0409236)
Ivanov E and Sutulin A 2004 *Phys. Rev.* D **70** 045022 (arXiv:hep-th/0403130)
Ivanov E and Sutulin A 1997 *Class. Quantum Grav.* **14** 843 (arXiv:hep-th/9604186)
Ivanov E and Sutulin A 1994 *Nucl. Phys.* B **432** 246
Ivanov E and Sutulin A 1997 *Nucl. Phys.* B **483** 531 (arXiv:hep-th/9404098) (erratum)
[5] Gates S J Jr, Grisaru M T, Roček M and Siegel W 1983 *Superspace* (Reading, MA: Addison-Wesley)
[6] Gates S J Jr 1997 Ectoplasm has no topology: the prelude *Int. Seminar on Supersymmetries and Quantum Symmetries (Dubna, Russia, 22-26 July)* pp 46–57 (arXiv:hep-th/9709104)
Gates S J Jr 1999 *Nucl. Phys.* B **541** 615 (arXiv:hep-th/9809056)
Gates S J Jr, Grisaru M T, Knutt-Wehlau M E and Siegel W 1998 *Phys. Lett.* B **421** 203 (arXiv:hep-th/9711151)
Kuzenko S M and tartaglino-Mazzucchelli G 2008 Different representations for the action principle in 4D, $\mathcal{N} = 2$ supergravity arXiv:hep-th/0812.3464